The Structure of Near-Minimum Edge Cuts

András A. Benczúr*
Department of Mathematics
M. I. T.

Abstract

Let G be an undirected k-edge connected graph. In this paper we give a representation for all edge cuts with capacity not exceeding roughly $\frac{6}{5}k$. This generalizes the *cactus representation* (Dinits *et al*) for all minimum cuts. Karger proved that the number of cuts within a multiplicative factor α of the connectivity is at most $O(n^{2\alpha})$. We improve this bound to $O(n^2)$ for $\alpha = 6/5$.

An important corollary of our result is a proof with new insights to the Lovász splitting theorem. A *splitting* of the edge pair us and vs at vertex s means replacing the two edges by uv. A splitting is *admissible* if it preserves the minimum local edge connectivity of the graph apart from vertex s. In other words, we may split us and vs if there is no set of degree minimum or minimum+1 containing both u and v; this property can be checked by our representation. Our new technique makes it possible to derive structure results of admissible pairs; among others we can show that splittable pairs form a connected graph unless the degree of s is odd or equals 4.

We believe that by using our representation it will be possible to improve on results using the cactus or splittings. One such result is the edge augmentation problem where one needs a minimum cardinality edge set which increases the connectivity of the graph. The algorithm of Naor, Gusfield and Martels applies the cactus representation to solve this problem; Frank's algorithm relies on the splitting theorems.

Key words: minimum cuts, edge connectivity, edge splitting-off, graph algorithms

*Address: Laboratory for Computer Science, Theory Group, 545 Technology Square, Cambridge MA 02139. E-mail: `benczur@theory.lcs.mit.edu`. Research supported in part by DARPA contract N00014-92-J-1799 and an NEC corporation grant. Part of this research was done while the author was student at Mathematical Institute, Hungarian Academy of Sciences, and Department of Computer Science, Eötvös University, Hungary, with the partial support from OTKA grants no. 1909, 7559 and 2118.

1 Introduction

Dinits *et al* [3] gave a concise *cactus representation* of all mincuts of an undirected graph. In this paper we extend this result to cuts with higher capacity. The following theorem of Lovász [13] (see also [14], Problem 6.53) and Mader [15] was our main motivation to investigate the structure of minimum and minimum+1 cuts. Assume $G = (V + s, E)$ is a graph with a distinguished vertex and we want to remove s from G by repeating the following step called *splitting* edge pairs off s: we pick a pair of edges us and vs, and replace them by a single edge uv. By the *splitting theorem* all edges can be split off s without decreasing the minimum local connectivity ω of G. It is easy to see that in order to preserve connectivity we may split an edge pair exactly if there is no set with capacity ω or $\omega + 1$ such that both edges enter this set.

Both the splitting theorem and the cactus representation is a useful tool in graph theory as well as in designing algorithms. Applications of the splitting theorem include composition rules for k-edge connected graphs [15] and the theorem of Nash-Williams for orienting $2k$-connected graphs to k-connected digraphs (Problem 6.54 in [14]). Cai and Sun [2] applied splittings for *augmenting undirected edge-connectivity* by adding optimum number of edges to the graph. Their ideas were improved by Frank [5] to the first polynomial time algorithm for this task. Another algorithm for the same problem (Naor *et al* [17]) uses the cactus representation as main tool. The recent fastest augmentation algorithms of Benczúr [1] and Gabow [8] are based on the idea of using the cactus, respectively the splitting theorems. It is open whether our representation can improve on any of these algorithms. One idea could be to accelerate algorithms by using our representation instead of the cactus. Another idea is that by understanding the structure of splittable pairs, we decide whether splitting is possible without using multiple edges. Note that it is not even known whether or not an optimal connectivity augmentation *without* using multiple edges can be found in polynomial time.

One improvement we make on earlier results is that we can describe the structure of splittable pairs. The theorems of Lovász and Mader prove only existence of such pairs. In Frank's augmentation algorithm [5] splittable pairs are found only by (quite time-consuming) flow algorithms. On the other hand the algorithm of Karger and Stein [10] suggests that finding the mincuts or even all near-minimum cuts might be fundamentally easier than to solve the maximum flow problem. By extending the cactus representation so as to include cuts of capacity ω and $(\omega+1)$ (where we assume that the minimum local connectivity ω is at least 6), we can derive the following *characterization of splittable pairs*. Let AS be the graph formed by edges of form uv such that (us, vs) is splittable. Then the complement of AS is the subgraph of either a cycle or two disjoint cliques connected by a path. An immediate consequence is that this graph is connected unless $d(s)$ is odd or equals 4. (Examples in Fig. 8 show that this last result is best possible.)

We prove a structure of minimum and minimum+1 cuts by excluding two simple configurations of such cuts: triangles in the sense of Gabow [7] and combs. It turns out that these configurations are forbidden among cuts within 6/5 times the connectivity as well. Hence with the same effort as what is needed to deal with the splittings, we can actually prove a stronger theorem. Since our representation is stronger than required, our version of the splitting theorem will also be slightly stronger: we show that there are always pairs which can be split so, that the capacity of no at most $(\omega+\epsilon)$-cut decreases. In this way $\epsilon/2$ *simultaneously* splittable pairs can be found. Comparing such edge pairs to splittable pairs in the original sense, if we split a pair, then another in the original (weaker) sense splittable pair may become non-splittable anymore: there may be a set of degree $\omega + 3$ which contains all four endvertices of the two edge pairs.

A further corollary of our result is that the number of cuts with capacity within a multiplicative

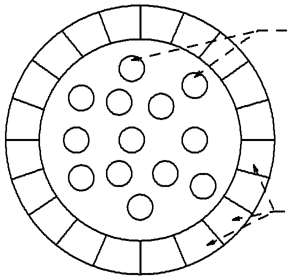

Fig. 1: The sunflower

factor $\alpha = 6/5$ of minimum is $O(n^2)$. From the cactus representation this bound is derived for minimum cuts; it is tight as the example of a cycle on n vertices shows. For this special α, our result is an improvement on the general result of Karger [9] showing that the number of cuts within an arbitrary factor $\alpha > 1$ is $O(n^{2\alpha})$. Very recently, Nagamochi *et al* [16] proved that the number of cuts within $\alpha = 4/3$ is also $O(n^2)$, but they do not have a structural result. For results concerning the number of cuts with (in some other sense) small capacities, see also [12] and [19]. Note that apart from the cactus theorem [3], ours is the only such result providing a structure of small cuts as well.

We review the basic properties of the *cactus representation* [3] of all mincuts of an undirected graph. A cactus is a graph which contains no cut edges and no two cycles with common edge. In other words a cactus is built up from a single vertex by recursively joining cycles to existing vertices. The cactus representation of a graph G is a cactus $\mathcal{K}$ such that a partition of $V(G)$ corresponds to $V(\mathcal{K})$. The mincuts of the cactus are precisely those which arise by erasing two edges of a cycle. Then the mincuts of G are precisely the edge sets of G connecting two components of a mincut of the cactus (as shown in the example of Fig. 2).

Now we describe how the above properties of the cactus are preserved or lost in our representation for near-minimum cuts. First it will be easy to generalize the tree-like structure of the cactus to our case. Then in most of this paper we deal with generalizing the concept of a cycle of the cactus representation, which is the "building block" of its tree-structure. The building block of our representation will be a "sunflower" (see Figs. 1 and 6). In a sunflower the vertex set of G is partitioned into two types of *atoms*; elements of one type are arranged in a cycle. Represented cuts have the following two properties. First, each cut divides the cycle of atoms into two consecutive parts just as in the cactus. Second, if we give a distribution on the cycle and there is a cut with this distribution, it is *unique*, i.e. there is only one way of arranging the remaining atoms to get a near-minimum cut. We are able to prove the $O(n^2)$ bound for the number of represented cuts because of this uniqueness. Note however that for a given partition of the cycle there is not necessarily a corresponding near-minimum cut.

Unfortunately, our proof for the representation does not yield an efficient algorithm. Can one extend, say, the cactus algorithm of Karzanov and Timofeev [11] to do it? There might be a hope for a fast algorithm, since the randomized algorithm of Karger and Stein [10] finds all cuts within capacity $6/5$ times the minimum in time $\tilde{O}(n^{2.4})$. But this is not all the bad news about the algorithmic aspects of our theorem. Assume that a sunflower is given: then one would expect that the represented cuts can be found in near constant time. But the uniqueness property is almost all we know about how to find a cut with a given distribution on the cycle around.

Note however that in order to prove the splitting theorem, we do prove something about the inside structure of the representation. Roughly speaking, what we prove is that edges leaving an atom are ordered in a cycle and edges of near-minimum cuts are consecutive in this order. We show that those edges incident to the extra vertex s can be split which are "far away enough" in this sense. The fact that near-minimum cuts behave so "regularly" around atoms shows that there is some kind of *planarity* in the representation. Unfortunately it is not true that by contracting atoms we get a planar graph; planarity could come into consideration only in some weaker way. We believe that such an idea could give a method of finding cuts in the representation, for example as finding shortest paths.

The rest of this paper is organized as follows. First we introduce a general tree-like representation for set systems, which is the "skeleton" of our representation as well as the cactus. In Section 3 we present lemmas describing allowed and forbidden configurations of near minimum cuts. In Section 4 these lemmas are extended to the case when a low degree node is present, as it is necessary in order to prove the splitting lemma. In Section 5 the main theorems describing the properties of cuts corresponding to the sunflower are proved. In Section 6 we prepare for the proof of the splitting theorem by defining a distance on edges such that far-enough edges cannot simultaneously be in the same near-minimum cut. Finally, in Section 7 we investigate the structure of admissible splittings.

2 Crossing sets and the cross graph

In this section we give a tree-like representation for general set systems. Though the representation contains only very rough information on the sets, it is a useful starting point of both our representation and the cactus. We shall use some basic notions regarding set systems, which we summarize here. For $X \subseteq V$, $\overline{X} = V - X$. Two sets C and D are called **intersecting** if neither of $C \cap D$, $C \cap \overline{D}$ and $\overline{C} \cap D$ is empty and **crossing** if in addition $\overline{C} \cap \overline{D} \neq \emptyset$. A set system is **laminar** if it contains no intersecting pair, **non-crossing** if it contains no two sets which are crossing. We use the well-known fact that sets of a laminar system can be desribed as nodes of a rooted tree.

Definition. For a set system $\mathcal{C} = \{C_1, \ldots, C_k\}$ we define the **cross graph** $\mathcal{G}(\mathcal{C})$, as follows. Let its vertex set be $\{1, \ldots, k\}$, let (i,j) be an edge iff C_i and C_j cross each other. Connected components of the cross graph will sometimes be called an (equivalence) **class** of $\mathcal{C}$ (under the relation of being connected). For a class we define its **atoms** as the roughest partition of the ground set where none of its sets is separated by a set of the class.

As an example, let us consider the graph of Fig 2. Sets of degree 6 form three equivalence classes with atoms F and $V(G) - F$; G, H, J and $V(G) - GHJ$ and finally ABC, D, E, K and $FGHJ$. Next we shall define the basic tree-structure; in Fig 2 it is also given for the above example.

Definition. Let T be a rooted tree with vertices labeled by subsets of $V(G)$. We say that T is the **tree structure of classes** if the following hold.

(i) $V(G)$ is the label of the root. The label of a vertex is the subset of any of its predecessors.

(ii) Vertices v on odd levels are in one-to-one correspondence with the classes of $\mathcal{G}(\mathcal{C})$ and

  (a) labels of the children of v are atoms of the corresponding class,

  (b) Unless v is the root, the complement of the label of v is also an atom.

Lemma 2.1. *No atom crosses a set of the system $\mathcal{C}$. No two atoms may cross each other.*

Proof. Let Z be an atom, assume $C \in \mathcal{C}$ crosses it. There are sets or complements of sets $C_1, \ldots, C_s$ of the class of Z with $\bigcap C_i = Z$. By definition no set of this class can cross Z. Hence C is not in this class and thus cannot cross any C_i. But then $Z \subseteq C_i$ is possible only if $C \subset C_i$ for all i. Thus

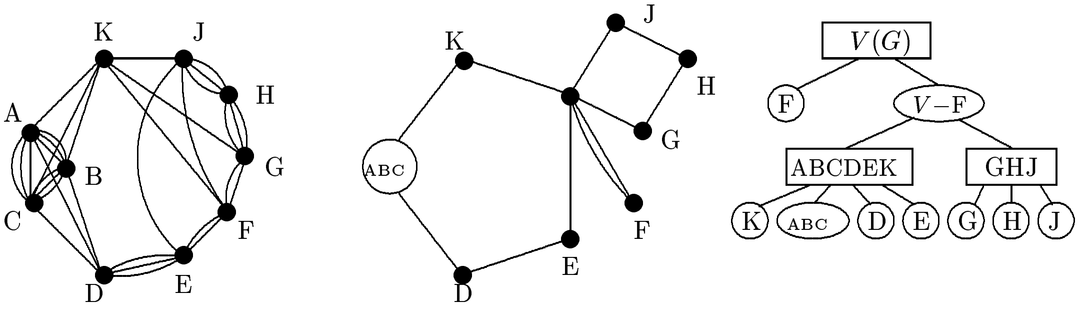

Fig. 2, *left*: a 6-edge-connected graph; middle: its cactus representation; *right*: the tree-structure of classes, which is a P–Q-tree in this case.

$C \subseteq \bigcap C_i = Z$ and C cannot cross Z. And as for the second part of the lemma, assume Z and Z' are two atoms crossing each other: then we repeat the above argument replacing C by Z'. □

Theorem 2. *There exists a tree structure of classes for each set system $\mathcal{C}$.*

Proof. By the previous lemma, atoms for a cross-free family. It is well-known that such a family can be turned laminar by complementing some of the sets. Now since atoms of a given class form a partition, at most one of them can be complemented in a laminar family. Thus atoms of a class are either the children of the root and form a partition (which may happen to one class), or they partition the one complemented atom. In both cases, atoms are all (immediate) children of a set in the system. □

Finally, we note that the cactus representation (Fig. 2) follows from this general framework. The **cactus representation** [3] represents all minimum cuts, or in other words all sets with minimum degree. Let us consider one class of the system of these sets. Such a class is described by the *cyclic partition lemma* [3] [18], which in our terms states that the atoms of the class have a natural cyclic order. The tree structure can thus be viewed as a P–Q-tree as follows. P-nodes correspond to even, Q-nodes to odd levels of the tree, hence to each class there is a corresponding Q-node. The fixed order of the children of a Q-node describes the *cyclic partition* of atoms. (Note that our condition (ii.b) slightly breaks the symmetry.) The P–Q-tree corresponding to mincuts can also be seen in Fig. 2. In a slightly different framework, the P–Q tree approach to the cactus representation is due to [12].

3 Properties of near minimum cuts

We begin to investigate connected components of the cross graph of near-minimum cuts. In the case of the cactus representation, crossing mincuts have a very simple structure, namely all of their four intersections have minimum degree as well. In Lemma 3.1 we actually show that this property extends to near-minimum cuts in some weaker sense. Also, we can derive that the degree of some intersections of near-minimum cuts is low and thus exclude some configurations (see Fig. 3). On the other hand there is one configuration (Fig. 5) which cannot arise for minimum cuts, but it is easy to construct such situation already for minimum and minimum+1 cuts (as for example in Fig. 6). To exclude the former configurations, we shall see that it suffices to impose the upper bound roughly $(6/5)\omega$ on the capacity of cuts.

Before starting the discussion we define some basic graph-theoretic concepts. Then we give the precise bound we require on the capacity of near-minimum cuts. We work with undirected multigraphs (which may contain parallel edges but no loops) $G = (V, E)$. For $X, Y \subseteq V$ we denote the set of edges leading from X to Y by $\delta(X, Y)$, $\delta(X) = \delta(X, \overline{X})$ for short, $d(X, Y) = |\delta(X, Y)|$, $d(X) = |\delta(X)|$. The submodular inequalities are used throughout the paper:

$$d(X) + d(Y) \geq d(X \cup Y) + d(X \cap Y) \ .$$

A **cut** is an edge set $\delta(C)$, it will also be denoted $(C|\overline{C})$. The **connectivity** of a graph is the minimum of the capacities of its cuts, in this paper it is denoted by ω. A system of cuts will usually be given as a set system on $V(G)$ consisting of one of the two sets C and $\overline{C}$ for each cut. For example, each non-crossing system can be turned to a laminar system by complementing some of its sets. Hence a non-crossing system of cuts can be considered as laminar, since we may select an appropriate representative component for each of its cuts.

Definition. Let $\Omega > \omega$ be a natural number. We shall call a cut Ω-near minimum (or **near minimum** short) if its capacity is not greater than the fixed $\Omega \geq \omega$. A proper subset C of V is Ω-extreme (or **extreme** short) if $\delta(C)$ is an Ω-near minimum cut. Throughout the paper we require the following (which implies $\omega \geq 6$ or $\omega = 4$) for Ω:

$$(*) \qquad \begin{cases} \Omega & \text{is even and } \omega > 5(\Omega - \omega) \quad \text{or} \\ \Omega & \text{is odd and } \omega > 5(\Omega - \omega) - 3. \end{cases}$$

Lemma 3.1. *Let C and D be two crossing extreme sets. Then $d(C \cap D, \overline{C} \cap D) \geq \mu$, where we define*

$$\mu = \lceil \omega - \Omega/2 \rceil \ .$$

Proof. We have

$$d(\overline{C} \cap D) \geq \omega = \Omega - (\Omega - \omega) \geq d(D) - (\Omega - \omega) \ ,$$

i.e.

$$d(C \cap D, \overline{C} \cap D) + \Omega - \omega \geq d(C \cap D, \overline{D}) \ .$$

Since

$$\omega \leq d(C \cap D) = d(C \cap D, \overline{C} \cap D) + d(C \cap D, \overline{D}) \ ,$$

$$\omega \leq 2d(C \cap D, \overline{C} \cap D) + \Omega - \omega \ ,$$

whence $\omega - \Omega/2 \leq d(C \cap D, \overline{C} \cap D)$ gives the result. □

3.1 Excluded configurations.

The following two lemmas give two forbidden configurations of extreme sets. They can be seen in Fig. 3. Actually, this figure contains three configurations. The first one is a comb with at least 3 teeth (Lemma 3.2). The second and third are configurations of 3 cuts with 7 or 8 non-empty parts. The third one is (an important) special case of the second: a triangle (3 pairwise intersecting sets with no common element) in the sense as defined by Gabow [7]. We also note that combs are the configurations excluded in interval systems; excluding combs and considering interval systems is the main idea of the cactus proof in Lehel *et al* [12].

Lemma 3.2. *Let C_1, C_2 and C_3 be three extreme sets. Then at least one of $C_1 \cap C_2 \cap C_3$, $\overline{C}_1 \cap \overline{C}_2 \cap C_3$, $C_1 \cap \overline{C}_2 \cap \overline{C}_3$ and $\overline{C}_1 \cap C_2 \cap \overline{C}_3$ is empty. In particular, if C_1, C_2 and C_3 are pairwise crossing, they divide the vertex set into six sets of form $C_i^{\epsilon_i} \cap C_j^{\epsilon_j}$ where $\epsilon_k = \pm 1$, $1 \leq i < j \leq 3$.*

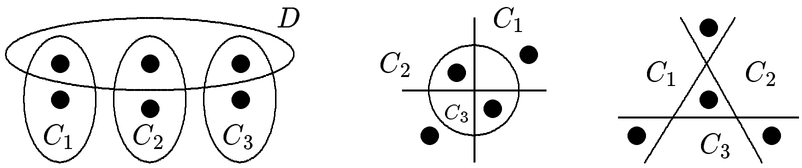

Fig. 3: the impossible configurations of Lemmas 3.3 and 3.2.

Proof. Assume the first part of the claim does not hold for C_1, C_2 and C_3. Then the edges leaving those sets described in the lemma are covered by edges contained in $\delta(C_i)$ for some i, furthermore with multiplicity two for those edges which connect two such sets. Thus

$$4\omega \leq d(C_1) + d(C_2) + d(C_3) \leq 3\Omega$$

contradicting $(*)$.

Now assume C_1 and C_2 cross each other. Let us consider the four parts $C_1^{\epsilon_1} \cap C_2^{\epsilon_2}$ of them. By possibly exchanging C_3 and $\overline{C}_3$ we may assume that $\overline{C}_3 \cap C_1 \cap \overline{C}_2$ is non-empty. Also without loss of generality we may assume that C_3 divides $C_1 \cap C_2$ and $\overline{C}_1 \cap \overline{C}_2$ into two non-empty parts, whence $C_1 \cap C_2 \cap C_3$ and $\overline{C}_1 \cap \overline{C}_2 \cap C_3$ are non-empty. Since our claim is true if C_3 does not divide $C_1 \cap \overline{C}_2$ and $\overline{C}_1 \cap C_2$, we may assume C_3 cuts $\overline{C}_1 \cap C_2$ and hence $\overline{C}_1 \cap C_2 \cap \overline{C}_3 \neq \emptyset$. But in this case we have built the four non-empty intersections contradicting the first part of the lemma. □

Lemma 3.3. *There are no three extreme sets C_1, C_2 and C_3 which cross a fixed set D with $C_i \cap D$ being disjoint.*

Proof. $\delta(C_i \cap D, \overline{D})$ are disjoint, and by Lemma 3.1 they contain at least μ–μ edges. But $\Omega \geq d(D) \geq 3\mu$ is in contradiction with $(*)$. □

Next we characterize extreme sets crossing a fixed set D. Informally, if some of these sets are crossing, they are pairwise crossing and form a "fiber"; these fibers are non-crossing and are ordered linearly by set inclusion (see Fig. 4). More formally, we state and shall use the following:

Lemma 3.4. *Let D be a fixed set. Then the system $\mathcal{D} = \{C \cap D : C$ is extreme and crosses $D\}$ is linearly ordered by set inclusion.*

Proof. We have to show that, as a subset system of D, $\mathcal{D}$ is a crossing-free family with no three disjoint sets. The first part is a consequence of Lemma 3.2 (there is no triangle of sets), while the second is Lemma 3.3 (there is no "comb"). □

3.2 A special configuration: polygons of sets.

There is a configuration, which is forbidden for minimum cuts (and in the cactus representation), but not for near-minimum cuts. This is a more general form of the third configuration of Fig. 3. That configuration can be viewed as a triangle with edges formed by the 3 sets. We define what we mean by polygons of more than 3 sets. (A pentagon can be seen in Fig. 5. The graph of Fig. 6 is an example when this configuration arises.)

Definition. A system of sets $C_1, \dots, C_k$, $k \geq 3$, is called a **polygon of sets**, if
(1) no set is contained in the union of the others,
(2) $\bigcup_{i=1}^{k} C_i \neq V(G)$,
(3) C_i crosses C_{i+1}, C_k crosses C_1, and no other pair of sets cross.

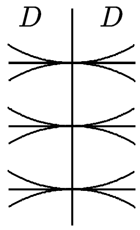

Fig. 4
Configuration of cuts crossing D

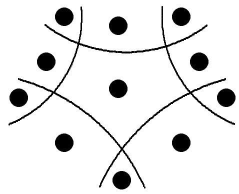

Fig. 5
Pentagon of sets

The **middle of the polygon** is $\bigcap_{i=1}^{k} \overline{C}_i \neq \emptyset$. This is a key notion in characterizing atoms; in the next lemma we show that being in the middle of a polygon is a property of *atoms* and not of particular cycles only.

Lemma 3.5. *Assume that the middle of a polygon is the union of some atoms. Then for each of these atoms there exists a polygon having that single atom in the middle.*

Proof. Consider a polygon $C_1, \ldots, C_t$ with smallest number of atoms in the middle containing a fixed atom Z. Unless $\bigcap \overline{C}_i = Z$, let $D \not\supset Z$ be a set which separates two atoms of this intersection and crosses some of C_i. Such sets exist, since the cuts defining the atoms have connected cross graph. First we note that if there are cuts C and C' of the polygon C_i, $i \leq t$, with $C \subset D$ and $C' \subset \overline{D}$, then there is another polygon of sets containing D. This polygon can be found as follows: start a walk from C' in both directions towards C around the polygon, and stop at the first sets crossing D. The sets visited by this walk, together with D, satisfy all requirements of polygons. The middle of this polygon contains Z but is strictly smaller than that of the polygon C_i, leading to a contradiction.

Now we try to find two sets C and C' with the above property. Let us pick a set C_i crossing D. First assume neither of its neighbors C_{i-1} and C_{i+1} crosses D. Neither D nor $\overline{D}$ can be a subset of a C_i, since both D and $\overline{D}$ contain atoms not in any set of the polygon. Hence by Lemma 3.4 C_{i-1} and C_{i+1} are contained by two different sides of D, giving the required two sets.

Now we may assume D crosses at least one of C_{i-1} and C_{i+1}, say C_{i-1}. C_i, C_{i-1} and D are pairwise crossing, by Lemma 3.2 we know how they divide the vertex set into six parts. There are two parts containing atoms from the middle of the polygon, these must be $D \cap \overline{C}_i \cap \overline{C}_{i-1}$ and $\overline{D} \cap \overline{C}_i \cap \overline{C}_{i-1}$. Let us consider C_{i+1}; by definition it is disjoint of all the atoms from the middle of the polygon. Also C_{i+1} does not cross C_{i-1} and is not contained in it by the definition of polygons. Hence by inspection we find that C_{i+1} is either contained in one side of D or crossing it. Furthermore, in the latter case we can see that C_{i+1}, D and C_i form a triangle contradicting Lemma 3.2. Repeating the above argument for C_{i-2} we get that C_{i+1} and C_{i-2} are the required two sets. □

4 The presence of a low degree node

In this section we extend results of Section 3 to the following case. Let $s \in V(G)$ be a distinguished vertex[1] of G. Let the *local* edge connectivity between each pair of vertices in $V(G) - s$ be at least

[1]Note that we do not distinguish s and $\{s\}$, we shall use $C + s$ and $C - s$ for union and difference with sets $C \subseteq V(G)$.

ω. But as a difference, we shall allow s to have degree *less than* ω. More precisely, we define

$$d_{\min} := 2\Omega - 4\mu = \begin{cases} 4(\Omega - \omega) & \text{if } \Omega \text{ is even} \\ 4(\Omega - \omega) - 2 & \text{if } \Omega \text{ is odd.} \end{cases}$$

and require

$$(**) \qquad d(s) \geq d_{\min}$$

Lemmas 4.1–4.4 are the counterparts of Lemmas 3.1–3.4 for the case when $d(s) < \omega$. Unless we give proofs, they are the same as for the original lemmas. There is no difference in the definition of polygons of sets (Subsection 3.2). However we distinguish one special polygon $(***)$, which encounters for most of the technical difficulties. In the last two lemmas modifications are necessary to handle this case.

$(***)$ $d(s) = d_{\min}$ and there is a polygon C_1, C_2, C_3, C_4 having s in the middle with $C_2 \cup C_4 = V(G) - s$.

Lemma 4.1. *Let C and D be two crossing near extreme sets with $C \cap D \neq s$. If $\overline{C} \cap D \neq s$, then $d(C \cap D, \overline{C} \cap D) \geq \mu$, and if $C \cap \overline{D} \neq s$, then $d(C \cap D, \overline{C}) \geq \mu$. In particular, $d(C \cap D, \overline{C}) \geq \mu$ whenever $C \cap D \neq s$.* □

Lemma 4.2. *The claim of Lemma 3.2 remains valid.*

Proof. We have to extend the proof to the case when one of the four intersections is s alone. In that case we get the inequality

$$3\omega + d_{\min} \leq d(C_1) + d(C_2) + d(C_3) \leq 3\Omega$$

contradicting $(*)$ and $(**)$. □

Lemma 4.3. *Lemma 3.3 remains valid unless $(***)$ holds and $C_i \cap D = s$ for some i. In particular, there are no three disjoint sets C_1, C_2 and C_3 which cross a fixed set D.*

Proof. First of all note that the proof of Lemma 3.3 remains valid unless $C_i \cap D = s$ for some i. Assume that C_1, C_2 and C_3 cross D and $C_i \cap D$ are pairwise disjoint. Then we may assume that $C_2 \cap D = s$. Also $d(C_2 \cap D) \leq \Omega - 2\mu = d_{\min}/2$. Furthermore, $C_i \cap \overline{D}$ cannot be disjoint since $s \notin \overline{D}$. Assume first that $C_i \cap \overline{D}$ and $C_j \cap \overline{D}$ intersect but are not equal. Then both $C_i - C_j$ and $C_j - C_i$ cross D. But by submodularity one of them is extreme, which yields another pair with disjoint intersection with $\overline{D}$.

Then we assume $C_1 \cap \overline{D} = C_2 \cap \overline{D}$. By Lemma 4.1

$$d(C_1 \cap D, C_1 \cap \overline{D}) = d(C_1 \cap D, C_2 \cap \overline{D}) \geq \mu$$

and similarly $d(C_2 \cap \overline{D}, \overline{C}_2 \cap \overline{D}) \geq \mu$, whence

$$d(C_2 \cap D, \overline{C}_2 \cap D) = d(s, D - s) \leq d(C_2) - 2\mu \leq d_{\min}/2$$

and in this case $(***)$ is satisfied, since C_1, $D - s$, C_3 and $\overline{D}$ form a polygon around s.

Finally assume $C_1 \cap \overline{D} = C_3 \cap \overline{D}$ is disjoint of $C_2 \cap \overline{D}$. Then $\delta(C_1 \cap \overline{D}, C_1 \cap D)$, $\delta(C_3 \cap \overline{D}, C_3 \cap D)$ and $\delta(C_2 \cap \overline{D}, D)$ are disjoint edge sets, each having at least μ elements. Hence we get the contradiction that $d(D) \geq 3\mu > \Omega$. □

Lemma 4.4. *Lemma 3.4 is valid if either $(***)$ does not hold or if we consider only those sets C crossing D for which $C \cap D \neq s$.* □

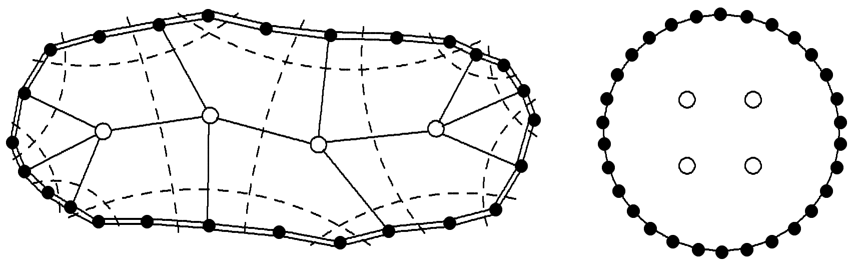

Fig. 6, *left*: 4-edge connected graph; *right*: its representation. Black nodes are *outside*, white ones are *inside* atoms; dashed curves indicate polygons of cuts around inside atoms.

5 One class of cuts: the sunflower

We have all the necessary tools to describe our representation for near-minimum cuts. We actually describe the structure of an equivalence class (connected component) of cut graph (Section 2). Thus to get the final representation, one can determine the tree structure of classes and the representation of each class can be assigned to the tree-vertex corresponding to the class.

Definition. Let us consider a class of the cross graph of near-minimum cuts. We call those atoms Z **inside atoms** for which a polygon of extreme sets exists with Z being in the middle of them. The rest of the atoms are called **outside atoms**. By a **sunflower** corresponding to the class we mean a cyclic order on the outside atoms, such that each cut of the class divides the cycle of outside atoms into two consecutive parts. (See Fig. 6 and 1.) We shall call such a division of outside atoms a **cut of the sunflower**.

The cycle of the cactus can be considered a special case of the sunflower with only outside atoms. But in the cactus, there is a further correspondence of a cycle and the minimum cuts, which is not included in the definition of the sunflower: if we cut a cycle into two, there is always a corresponding mincut of the graph. In general we do not have such a simple property for near-minimum cuts for two reasons. First, given a cut of the sunflower, there may not be any corresponding near-minimum cut of the graph. Second, if such cuts exist, we do not know the distribution of *inside* atoms. In the example of Fig. 6 both of these cases can be seen.

Unfortunately we cannot present a procedure to find the near-minimum cut (i.e. the distribution of the inside atoms) corresponding to a cut of the sunflower or even to decide whether such cut exist. However, what we next show is that if such cut exist, it is unique. This is not really true: if $(***)$ holds, both a set D and $D+s$ are extreme and they cut the sunflower in the same way, since s is inside atom. But as the theorem shows, this is the only exceptional case.

Theorem 5.1. *Let C_1 and C_2 be two sets of the same equivalence class. The $C_1 \cap C_2$ is either empty, or contains at least one outside atom, or $C_1 \cap C_2 = s$ and $(***)$ holds.*

Proof. Assume $C_1 \cap C_2$ contains only inside atoms. Applying submodularity, we may assume $C_1 \cup C_2 = V(G)$. We shall also assume minimality, more precisely that $C_1 \cap C_2$ is minimal with respect to

- $C_1 \cup C_2 = V(G)$,
- $C_1 \cap C_2 \neq s$ and
- $C_1 \cap C_2$ does not contain outside atoms.

When $C_1 \cap C_2 = s$, $C_1 \cup C_2 = V(G)$ and s is an inside atom, by definition $(***)$ holds. Hence the above conditions cover all possible cases when the theorem could fail.

We aim to find two sets C^+ and C^- crossing both C_1 and C_2 with $C^+ \supset C_1 \cap C_2$, $C^- \cap C_1 \cap C_2 = \emptyset$. Assume we have such sets. Then to complete the proof of the theorem, without loss of generality we assume $d(C_1 \cap C_2, \overline{C}_1) \geq d(C_1 \cap C_2, \overline{C}_2)$. Hence $d(C_1 \cap C_2, \overline{C}_1) \geq \omega/2$. Now

$$d(C_1) = d(C_1 \cap \overline{C^+}, \overline{C}_1) + d(C_1 \cap C^-, \overline{C}_1) + \\ + d(C_1 \cap C_2, \overline{C}_1) \geq 2\mu + \omega/2 \ ,$$

which is greater than Ω by $(*)$. This proves the theorem. In the next sequence of claims first we show that for each atom in $C_1 \cap C_2$ there are sets "above" and "below" it crossing both C_1 and C_2. Then we complete the proof by considering all such sets for the set of atoms in $C_1 \cap C_2$ and picking the "topmost" C^+ and "lowest" C^- among them.

Claim. *(i) There is no extreme set C' such that $C' \cap C_1$ is a proper subset of $C_1 \cap C_2$. If $s \in C_1 \cap C_2$, neither $C_1 - s$ nor $C_2 - s$ is extreme.*

Proof. If $C_1 - s$ is extreme, $C_1 - s$ and C_2 are in contradiction with the minimality of $C_1 \cap C_2$. Now we may assume $C' \cap C_1 \neq s$, whence the first part of the claim can also be proved by minimality. □

Claim. *(ii) There is no extreme set C of the considered class with either $C \subseteq C_1 \cap C_2$ or $C_1 \subset C \subset \overline{C}_2$ (the latter with strict inclusions).*

Proof. Let C contradict the first part of the claim. There must be another set C' of the class crossing C. Hence $\emptyset \neq C \cap C'$ and $\emptyset \neq C \cap \overline{C}'$ are proper subsets of $C_1 \cap C_2$. At least one of them is not s alone, contradicting the minimality of $C_1 \cap C_2$. And if C contradicts the second part, one of the pairs C_1 and $\overline{C}$ or $\overline{C}_2$ and C give contradiction with (i). □

Let us fix an atom $Z \subseteq C_1 \cap C_2$. Let $D_1, \ldots, D_k$ be a polygon of sets with $Z = \bigcap_{i=1}^k \overline{D}_i$.

Claim. *(iii) Let D be a set of the polygon which is not the proper subset of $\overline{C}_1$ or $\overline{C}_2$. Then D crosses at least one of C_1 and C_2.*

Proof. Assume D does not cross C_1 and C_2. Then there are two possibilities for D: D or $\overline{D}$ must be the subset of either $\overline{C}_1$ or $\overline{C}_2$, or it must be a set as described in (ii). $\overline{D} \subseteq C_i$ is impossible by $Z \not\subseteq D$, $D \not\subseteq C_i$ by assumption and the last case contradicts (ii). □

Claim. *(iv) If a set D of the polygon crosses C_1 but not C_2, then $\overline{D} \subseteq C_2$.*

Proof. If D crosses C_1, both $D \cap C_2$ and $\overline{D} \cap C_2$ are non-empty. Then since D does not cross C_2, either $D \subseteq C_2$ or $\overline{D} \subseteq C_2$. But in the former case $D \cap C_1 \subseteq C_1 \cap C_2 - Z$, contradicting (i). □

Claim. *(v) A set of the polygon which is not the proper subset of $\overline{C}_1$ or $\overline{C}_2$ crosses both C_1 and C_2. Furthermore, there are at least two disjoint such sets.*

Proof. Assume there is a set D crossing C_1 but not C_2. Let D' be another set of the polygon disjoint of D. (Such set exists since there is no triangle of sets.) There are three possibilities: (1) D' may cross C_2, (2) D' may cross C_1 but not C_2, or (3) $D' = C_2$. $D' \subset \overline{D} \subseteq C_2$ by (iv) and since D and D' are disjoint. Hence case (1) and (3) are impossible. Finally, in case (2) (iv) holds for D': $\overline{D}' \subseteq C_2$, hence $D \cap D' \supseteq \overline{C}_2$, contradicting that D and D' are disjoint.

Finally we prove that there are two disjoint sets crossing both C_1 and C_2. There is one such set D, since otherwise by (iii) all sets are inside $\overline{C}_1$ or $\overline{C}_2$, whence the middle of the polygon contains not only Z. Assume all other sets of the polygon crossing C_1 or C_2 also cross D. The possible such sets are only the two neighboring set of D in the polygon. Then the remaining sets are inside $\overline{C}_1$ or $\overline{C}_2$ and it is easy to see that they cannot form a polygon around Z. □

Now consider all atoms in $C_1 \cap C_2$. Define the polygons around them as we did it above. Let us consider those sets of them which are crossing both C_1 and C_2. By (i) and Lemma 4.4 we may linearly order their intersection with C_1 by set inclusion. Let then C^- and C^+ be the smallest and the largest among them, $C^- \subset C^+$.

We complete the proof by showing that $C^- \cap C_1 \cap C_2 = \emptyset$ and $C^+ \supset C_1 \cap C_2$. If this claim fails, there are atoms in the "forbidden" parts. Let us assume Z is an atom in $C^- \cap C_1 \cap C_2$ such that there is a polygon around Z containing C^-. By (v) this polygon has another set D crossing both C_1 and C_2 but not C^-. Because of the minimality of C^- in the linear order, D or $\overline{D}$ contains C^-. On the other hand $D \subseteq \overline{C^-} \cup Z$, since C^- and D are two disjoint members of a polygon around Z. This contradicts that D or $\overline{D}$ contains C^-. The argument is the same for $\overline{C^+}$. □□□

Theorem 5.2. *Each connected component of $\mathcal{G}$ can be represented by a sunflower.*

Proof. Let $\mathcal{D}$ be an arbitrary connected system of extreme sets. We can define polygons and inside–outside atoms with respect to $\mathcal{D}$. We prove by induction on $|\mathcal{D}|$ that outside atoms are ordered in a cycle such that each $D \in \mathcal{D}$ contains a consecutive sequence of them. In the proof we shall assume this above property for a system $\mathcal{D}$ and consider another connected system $\mathcal{D} + C$ of extreme sets.

There are two possible changes in the set of outside atoms. First, some atoms Z may be cut into two (if both $Z \cap C$ and $C \cap \overline{Z}$ are non-empty). Second, some of the old outside atoms may turn out to be in the middle of a polygon of sets (this polygon must contain C). We update the cyclic order of outside atoms, as follows. First if an atom is cut into two, we consider them tied and allow to break ties as necessary. Second, after this step we simply remove (new) inside atoms without changing the order anywhere else. Clearly, sets of $\mathcal{D}$ still contain a consecutive sequence of (new) outside atoms. In the sequel we prove the same for C.

Let us call a set which is the union of atoms **consecutive** if it contains a consecutive sequence of outside atoms. Let the two **boundaries** of a consecutive set be the two sidemost outside atoms contained by them. Let $D_1, \ldots, D_t$ be those $D \in \mathcal{D}$ which cross C and for which $D_i \cap C \neq s$ in case (∗∗∗) holds. Assume $D_i \cap C$ is an increasing set system (as proved in Lemma 4.4). Furthermore, let D_1' be that D_i for which $D_i \cap \overline{C}$ is smallest. (Note that either $D_1' = D_1$ or they cross each other). Let us call D_i (and $\overline{D}_i$) **good** if both $C \cap D_i$ and $\overline{C} \cap D_i$ are consecutive (respectively, if the same holds replacing D_i by $\overline{D}_i$).

If our claim holds, all D_i and $\overline{D}_i$ are good. Assume we know that D_t and $\overline{D}_t$ are good. Let us fix the four consecutive sets of form $D_t^{\epsilon_1} \cap C^{\epsilon_2}$, $\epsilon_i = \pm 1$. There is one unique way of arranging the given sets such that C is not consecutive. Assume this is the case and let us pick a cut of $\mathcal{D}$ crossing D_t. It is consecutive, but it cannot be placed in the given setting. This gives the claim of the theorem. Through the next claims we prove this for each D_i. First we build an inductive machinery which enables us to infer that D_{i+1} is good, provided D_i is good. Here the basic idea is to connect D_i by *consecutive* sets.

Claim. *In the following cases D_i is good. (i) $D_i \cap C$ is consecutive and a $C' \subset C$, $C' \neq C$ crosses D_i. (ii) D_i and D_j cross each other and D_j is good.*

Proof. First we prove that if $D_i \cap C$ is consecutive and $C' \in \mathcal{D}$ is such that it crosses D_i and $D_i \cap C \cap C'$ contains outside atoms, then D_i is good. To prove, note that since D_i is consecutive, the only possibility for D_i not being good is if it has no common boundary with $D_i \cap C$. But then by the assumption on $D_i \cap C \cap C'$, C' could not be consecutive.

Case (i) is exactly the previous paragraph if we note that $C' \cap D_i \subset D_i \cap C$ contains outside atoms by Theorem 5.1. And as for (ii), we may consider the six components of C, D_i and D_j allowed in Lemma 4.2 and we may assume that $C \cap D_i \subset C \cap D_j$ and $\overline{C} \cap D_j \subset \overline{C} \cap D_i$. Then the claim of the previous paragraph can be applied again: C' can be selected to be D_j and $D_i \cap C$ is consecutive since it is the intersection of D_i and $C \cap D_j$. □

Claim. *In the following cases D_i is good as well. (iii) $D_j \subset D_i$ is good and there is a path of the cross graph connecting D_i and D_j such that all interior extreme sets of the path are the subset of C. (iv) If D_j is good for some $i > j$.*

Proof. To prove (iii), by (i) it suffices to show that $D_i \cap C$ is consecutive. Let $C_1, \dots, C_t$ be the path connecting D_i and D_j. Then by induction on k we may prove that $(D_i \cap C) \cup \bigcup_{\ell=1}^{k} C_\ell$ is consecutive. For $k = t$ it proves that $D_i \cap C$ is consecutive.

And for (iv) it suffices to show that D_{j+1} is consecutive. This holds by (ii) if D_{j+1} crosses D_j. Otherwise let $C_1, \dots, C_t$ be the shortest path connecting D_{j+1} and D_j in the cross graph. If for all k $C_k \subset C$, we are done by (iii). If $\overline{C}_k \subset C$, we may replace C_k by $\overline{C}_k$. And if for all k, $C_k \subseteq \overline{C}$ (or $\overline{C}_k \subseteq \overline{C}$), we may use (iii) again with replacing C for $\overline{C}$. Hence the only remaining case is when there is a C_k crossing C. Let then $C_k = D_\ell$. Since either $D_\ell \cap C \subset D_j \cap C$ or $D_\ell \cap C \supset D_{j+1} \cap C$, we are in contradiction with the path being shortest. □

Now we prove the initial step of the induction, i.e. that D_1, D'_1 and $\overline{D}_t$ are good. Let X, respectively Y be the sets of outside atoms in $D_1 \cap C$ and $D_1 \cap \overline{C}$. We define $C_0 = D_1$, $C_1, C_2, \dots$ and $C'_0 = D_1$, $C'_1 = D'_1$, $C'_2, C'_3, \dots$ as paths of $\mathcal{G}(\mathcal{D})$ for which $C_i \subset C$, $C'_i \subset \overline{C}$ for $i \geq 1$ and which contain the largest number of atoms of X, respectively Y. Let these sets of atoms be X' and Y'. Let $Z = X \cup Y - (X' \cup Y')$. Since C_i and C'_i are consecutive, X' and Y' are also consecutive. Next we show that there is at most one *old* atom not contained in X' and Y'; then it will be possible to show that this single atom is in the "correct" place.

Claim. *(v) $|Z| \leq 2$ and it is exactly two only if it consists of two atoms whose union is an atom of $\mathcal{D}$ (old atom). (vi) If $|X| > 1$, there exists a $C' \in \mathcal{D}$, $C' \subset C$ crossing D_1.*

Proof. Assume (v) does not hold: then there is a $C' \in \mathcal{D}$ separating two atoms of Z. C' cannot cross C by the minimality of $D_1 \cap C$ and $D'_1 \cap \overline{C}$. Hence either $C' \subset C$ or $C' \subset \overline{C}$. Assume w.l.o.g. that the first case holds. The shortest path of $\mathcal{G}(\mathcal{D})$ connecting C' to the last C_i must have all sets inside C, again by the minimality of $D_1 \cap D$. This path added to the path C_i contradicts the maximality of the path C_i.

And to show (vi) assume $|X| > 1$ an use similar argument as for (v): pick a C'' separating two atoms of X and a path of $\mathcal{G}(\mathcal{D})$ connecting C'' to D_1 inside C. Then this path contains the required C'. □

Claim. *(vii) D_1 and $\overline{D}_t$ are good.*

Proof. By (ii) D_1 is good iff D'_1 is good. By replacing each D_i to its complement, we can also show that $\overline{D}_t$ is good. Hence we prove only that D_1 (or D'_1) is good. We are done if $X = X'$ and $|X| > 1$ (or $Y = Y'$ and $|Y| > 1$) since then the conditions of (i) for D_1 (or for D'_1) are satisfied: $D_1 \cap C$ is consecutive and there is a cut crossing D_1 by (vi). We are trivially done if $|X| = |Y| = 1$.

By (v) $|Z| \leq 2$. We distinguish cases concerning $|Z|$. If $|Z| = 0$, we are done with the proof since the requirements of (i) hold as above. If $|Z| = 1$, the only case when the conditions of (i) are not met is when $X = X'$, $|X| = 1$ (or the case is similar with Y). Pick then a cut D separating the two atoms X and Z. If $D \subset C$, D crosses D_1 and conditions of (i) are met again. And if $D \subset \overline{C}$, D must cross one set of the sequence C'_i defining Y. Hence we get a contradiction with the maximality of Y.

The remaining case is when $|Z| = 2$. Then the consequences of (v) hold. In this case if $|X| = 1$, $D_1 \cap \overline{D}$ contains an atom which is equivalent to X in the cyclic order. Hence $D_1 \cap \overline{D}$ contains exactly the same outside atoms of the order as D_1, D_1 is good. The same proof works if $|Y| = 1$.

Finally let $|X|, |Y| > 1$, when by (vi) there exists a $C' \subset C$ and a $C'' \subset \overline{C}$ crossing D_1, respectively D_1'. Hence if $D_1 \cap D$ or $D_1' \cap \overline{D}$ is consecutive, we are done by (i). $D_1 \cap D_1'$ is consecutive and it consists of three consecutive sequences: X', Y' and Z. But there is no arrangement of these sets such that neither of $D_1 \cap D$ or $D_1' \cap \overline{D}$ is consecutive. Hence we completed showing that D_1 is good. □□□

Theorem 5.3. *There are $O(n^2)$ near minimum cuts of a graph.*

Proof. There are at most $O(n)$ atoms, by Theorem 5.1 this implies the $O(n^2)$ bound. □

6 The distance of edges leaving a fixed atom

This section is a preparation for proving splitting theorems. We aim to characterize edges leaving a fixed atom as being ordered "around" it (this atom will contain the extra vertex s in the applications). By the end of this section, we shall be able to define a *distance* between such edges and prove that "far-enough" edges cannot be simultaneously cut by an extreme set. The fact that such distance can be defined shows that there is some kind of *planarity* in the representation. However, as we shall see, the distance relates only equivalence classes of edges and it is unclear what happens within a class: some edges may "jump" to farther atoms, thus breaking planarity. Also note that the simplest way of using planarity could be to show that by contracting atoms we get a planar graph, however this claim is false.

First we show that Theorems 5.1 and 5.2 implicitly describe *extreme sets* as being "around" atoms. First we fix an atom Z and consider those sets not containing Z; thus we choose one side of each cut. Then we may rephrase Theorem 5.1 as follows: for each near-minimum cut there is a corresponding consecutive sequence, a *segment*, of outside atoms, which uniquely describes C. We define the **segment order** of extreme sets by ordering them according to the (clockwise) first outside atom contained by this segment, and break ties according to the last such atom. For inside atoms, it is easy to see that this order is the same as the order of sets in a particular polygon around them. Also, now we can talk about sets "around" *outside* atoms. Now we use the segment order to define ordering of edges.

Definition. Let us fix an atom Z. Let us consider those extreme sets which are in the class of Z but does not contain Z. Let $\mathcal{C}[Z]$ consist of the maximal such sets. (Intuitively, elements of $\mathcal{C}[Z]$ are "near" Z.) Let us divide $\delta(Z)$ into equivalence classes: two *edges* are *equivalent* if for all sets of $\mathcal{C}[Z]$ either both or none of them is cut by the set. Let $[e]$ denote the class of $e \in \delta(Z)$. Let $\mathcal{C}[e]$ be the subset of $\mathcal{C}[Z]$ consisting of sets cutting e. Let $C^-[e]$ and $C^+[e]$ denote the first and last set of $\mathcal{C}[e]$ in the segment order. Then we can order classes $[e]$ according to the segment order of $C^+[e'] \cap C^-[e']$.

Lemma 6.1. *Let Z be an atom. Then each set $C \in \mathcal{C}[Z]$ cuts a consecutive sequence of $\delta(Z)$.*

Proof. If the claim of the lemma fails, there are four edges f, e, f' and $e' \in \delta(Z)$ arranged clockwise such that D cuts f and f' but not e and e'. Let X and X' be the set of outside atoms of $C^+[e] \cap C^-[e]$, respectively $C^+[e'] \cap C^-[e']$. By maximality of sets in $\mathcal{C}[Z]$, the segment order on $\mathcal{C}[Z]$ is the same as the order according to the first outside atom. Hence D must entirely contain one of X and X', say X, furthermore both $C^+[e]$ and $C^-[e]$ must cross D.

Assume first $C^+[e] = C^-[e] = C$. Then $C \cap \overline{D}$ does not contain outside atoms, since $X \subset D$. Hence by Theorem 5.1 (***) holds, $C \cap \overline{D} = s$ and $D \cup s$ is an extreme set contradicting the maximality of D.

If the above case does not hold, $C^+[e]$, $C^-[e]$ and D pairwise cross each other. $C^+[e] \cap C^-[e] \cap \overline{D}$ contains an endvertex of e but no outside atoms, since $X \subset D$. But by Lemma 4.2 there are six non-empty components of D, $C^+[e]$ and $C^-[e]$, which arise as intersections of two of these sets or complements. Hence $C^+ \cap C^- \cap \overline{D}$ is of this form and by Theorem 5.1 it should contain outside atoms. □

Lemma 6.2. *Assume Z is an outside atom. Then there are two edges and two outside atoms which are incident to each other in their corresponding order, but can never be cut, respectively contained simultaneously by any set of $\mathcal{C}[Z]$. Hence both the segment order on sets of $\mathcal{C}[Z]$ and the order on classes $[e]$ of $\delta(Z)$ are* linear.

Proof. If there is no such pair of edges or outside atoms, there exists an extreme set for each consecutive pair which cuts, respectively contains both of them. Let $\mathcal{C}$ be a minimal subsystem of $\mathcal{C}[Z]$ which has this property; these sets form a circle of cuts containing Z in the middle. □

Definition. Let e_1 and $e_2 \in \delta(Z)$. If Z is an outside atom, the ordering of $\delta(Z)$ is linear; let m edges be between $[e_1]$ and $[e_2]$. And if Z is an inside atom, the order is cyclic and let m be the number of edges on the shorter segment between $[e_1]$ and $[e_2]$. Then the **distance** of e_1 and e_2 is defined to be $m + |[e_1]| + |[e_2]|$.

Lemma 6.3. *Let Z be an atom. Let e, f be an edge pair of distance at least $d_{\min}/2 + 1$ in $\delta(Z)$. Assume an extreme set $C \supset Z$ cuts both e and f. Then Z is an outside atom and C is contained by the first or last set in the linear order of $\mathcal{C}[Z]$.*

Proof. Let us consider $\mathcal{C}[Z]$. If $C \notin \mathcal{C}[Z]$, there is a set of $\mathcal{C}[Z]$ containing C (if we assume $Z \not\subset C$). That set also cuts e and f, we may replace C by that set. Then by Lemma 6.1 we may assume C cuts a consecutive sequence of edges, hence $d(C, s) \geq d_{\min}/2 + 1$. Now let us consider the sets C^+ and C^- which are next to C in the order of $\mathcal{C}[Z]$. Unless Z is outside atom and C is the first or last set of $\mathcal{C}[Z]$ (which is precisely the special case described in the lemma), C crosses both C^+ and C^-. Since there is no triangle of sets, $\overline{C} \cap C^+ \cap C^- = \emptyset$. Thus

$$\begin{aligned} d(C) \geq d(C, \overline{C} \cap C^+) + d(C, \overline{C} \cap C^-) + d(C, s) \geq \\ \geq 2\mu + d_{\min}/2 + 1 > \Omega \end{aligned}$$

by Lemma 4.1 and (∗∗), a contradiction. □

7 The Splitting Theorem

In this section we give a new proof for Lovász' Splitting Lemma [13], [15] in a slightly generalized form. **Splitting** a pair of edges incident to s, us and vs, say, means replacing us and vs by a single edge uv. In Lovász' Splitting Theorem it is proved that there exists an edge pair such that the connectivity of $G - s$ does not decrease after splitting that pair. In that context, such a splitting is called admissible. In the sequel we define the splitting of edges us and vs Ω-admissible (or **admissible** for short) if the capacity of no Ω-near minimum cut decreases after splitting it. We prove that there exists an Ω-admissible pair whenever (∗) and (∗∗) hold for the minimum connectivity and the degree of s. Provided $\Omega = \omega + 1$ (which is the original form of the Splitting Theorem), our condition on $d(s)$ is the same as in earlier results (which is the necessary condition [4]). Our theorem is weaker since we require the connectivity to be at least 6.

Mader [15] (and later Frank [4]) has a strengthening of Lovász' result [13] by showing that there is a splitting which preserves the local connectivity between *all* pairs of nodes (not only the global

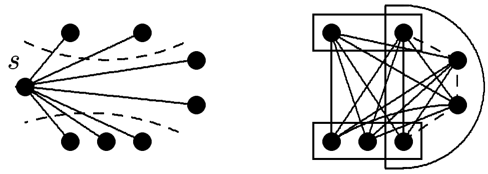

Fig. 7, *left*: s is an outside atom; edges incident to s are in the order defined in Section 6. *Right*: AS consists of two cliques, dashed edges may belong to AS if ω is odd.

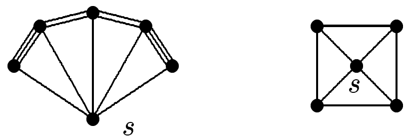

Fig. 8.: Examples when AS is not connected. *Left*: AS is a $K_{2,2}$ and an independent vertex. *Right*: AS consists of 2 independent edges.

minimum), provided $d(s) \geq 4$ and there is no cut edge incident to s. Notice that an Ω-admissible splitting preserves local connectivity up to $\Omega - 1$, but not all connectivities. However, in another sense our theorem is stronger than Mader's. There may be several splittable pairs in Mader's sense, but after we have split such a pair, we do not know whether we may or not split another one of them: there may be a set of degree $\omega + 3$ which contains all four endvertices. But in our theorem we may split $\lceil (\Omega - \omega)/2 \rceil$ pairs *simultaneously* since these pairs avoid *all* near minimum cuts. Hence our theorem is a good counterpart to the results of [15]–[4].

The main new part of our result is a characterization of the deeper structure of Ω-splittable pairs. We define the **graph of admissible splittings** AS as follows. Let $V(AS)$ be the set of vertices adjacent to s, let two such vertex be connected by an edge iff the splitting of the two adjacent edges is admissible. In the next theorem we describe AS; in the discussion we distinguish basically three cases depending on the characterization of minimal atoms containing s. We describe AS in detail only for $\Omega = \omega + 1$, in general AS has similar structure but is sparser. Among others we show that if $\Omega = \omega + 1$, then apart from marginal cases AS is connected.

Theorem 7. *There exists an Ω-admissible pair in graphs satisfying $(*)$ and $(**)$. For $\Omega = \omega + 1$,*

(i) if s is not an atom alone for any class, AS contains a complete k-partite graph for some $k \geq 2$;

(ii) if s alone is an inside atom for some class, then $\overline{AS}$ is empty for ω even and the subgraph of a Hamiltonian cycle for ω odd (Fig. 9);

(iii) finally if s is an outside atom, $\overline{AS}$ consists of two vertex disjoint cliques if both $d(s)$ and ω are even (Fig. 7). If either of them is odd, at most one of the following two possibilities can happen:

(a) if $d(s)$ is odd, the two cliques may have a common vertex, and

(b) if ω is odd, $\overline{AS}$ may also contain some edges of a path connecting the two cliques.

In particular, AS is not connected only if either $d(s)$ is odd or ω is odd and $d(s) = 4$. This result is best possible, as the examples of Fig. 8 show.

Proof. Let $\mathcal{C}'$ be the system of atoms containing s for each equivalence class. Let $\mathcal{C}$ be the set of (inclusionwise) minimal elements of $\mathcal{C}'$. By the non-crossing property of atoms (Lemma 2.1), the complements of elements of $\mathcal{C}$, $\{C_1, \ldots, C_t\}$ are disjoint. Unless s is an atom alone, we are done (both in the general and the $\omega + 1$ case) by the next lemma.

Lemma 7.1. *Let $\mathcal{C}$ be defined as above. Then we can split us and vs either if u or $v \notin \bigcup C_i$, or if u and v are in distinct C_i.*

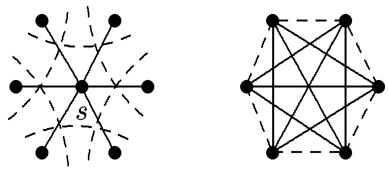

Fig. 9, *left:* s is an inside atom of a sunflower, edges incident to s are in the order as defined in Section 6; *right*: AS contains the complement of a cycle (dashed edges are not necessarily in AS).

Proof. To show that us and vs are not splittable, we have to give an extreme set D with $u, v \notin D$ and $s \in D$. Since C_i, $i \leq t$, are the complements of minimal atoms containing s, there must be an i such that $D \subseteq C_i$. This is impossible in both cases of the lemma. □

Now we may assume that s is an atom. By the tree structure of classes, Theorem 2, the class having this atom is unique. Assume s is an inside atom. (See Fig. 9.) Then there exists a pair of edges as required in Lemma 6.3 since $d(s) \geq d_{\min}$. If $\Omega = \omega + 1$, $d_{\min} = 4$ if ω is odd and $d_{\min} = 2$ if ω is even. Hence by Lemma 6.3, the only non-splittable pairs are those of distance 2, (neighboring edges) in the first case, while in the second case all pairs are splittable.

Now assume s is an outside atom. (See Fig. 7.) Then by Lemma 6.2 the first and last edges of the linear order of $\delta(s)$ are splittable. Let $\Omega = \omega + 1$. By Lemma 6.2 there is a first set C and a last set C' of $\mathcal{C}[s]$. If an edge pair is not cut by either C or C', then they can be non-splittable only if ω is odd and they are neighboring, by Lemma 6.3. We complete the discussion by proving that $d(s, C \cap C')$ is 0 if $d(s)$ is even and at most one if odd. It suffices to show that $d(C, s) \leq \lceil d(s)/2 \rceil$. Assume not: then since $C + s \neq V(G)$, $\omega \leq d(C + s) < d(C) - 1 \leq \Omega - 1 = \omega$, a contradiction. □

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